

# On the Cover-Hart Inequality: What's a Sample of Size One Worth?

Tilmann Gneiting, Universität Heidelberg

March 8, 2013

## Abstract

Bob predicts a future observation based on a sample of size one. Alice can draw a sample of any size before issuing her prediction. How much better can she do than Bob? Perhaps surprisingly, under a large class of loss functions, which we refer to as the Cover-Hart family, the best Alice can do is to halve Bob's risk. In this sense, half the information in an infinite sample is contained in a sample of size one. The Cover-Hart family is a convex cone that includes metrics and negative definite functions, subject to slight regularity conditions. These results may help explain the small relative differences in empirical performance measures in applied classification and forecasting problems, as well as the success of reasoning and learning by analogy in general, and nearest neighbor techniques in particular.

*Key words:* Bayes risk; decision theory; kernel score; loss function; metric; nearest neighbor; negative definite; proper scoring rule.

## 1 Introduction

Alice and Bob compete in a game of prediction. The task is to predict a future observation, such as a class label, or a real-valued, vector-valued or highly structured outcome. Before issuing a point forecast, Alice and Bob may sample from the underlying population. Bob has access to a sample of size one only, whereas Alice can draw a sample of any desired size. The predictive performance is evaluated by means of a loss function,  $L(y, y') \geq 0$ , where  $y$  is the point forecast, and  $y'$  is the realizing value of the future observation,  $Y'$ . Intuitively, we expect Alice to do much better than Bob, as she can gather essentially all information available, thereby attaining or approximating the Bayes risk, namely

$$\alpha \equiv \inf_y \mathbb{E}_P L(y, Y'),$$

where  $Y'$  has distribution  $P$ . However, Cover and Hart (1967) and Cover (1968) showed that under misclassification loss and squared error, Bob's risk,  $\beta$ , is at most twice Alice's risk,  $\alpha$ , that is,

$$\alpha \leq \beta \equiv \mathbb{E}_P L(Y, Y') \leq 2\alpha, \quad (1)$$

where  $Y$  and  $Y'$  are independent with distribution  $P$ .

In an elegant and thought-provoking discussion, Cover (1977) noted that the inequality continues to hold if the loss function is a metric. In this paper, we seek a unifying treatment of these remarkable facts. Section 2 identifies large classes of loss functions that satisfy the Cover-Hart inequality (1), including both metrics and negative definite functions. Section 3 considers probabilistic predictions, where the forecasts take the form of predictive probability distributions, and the predictive performance is evaluated by means of a proper scoring rule,  $S(Q, y')$ , where  $Q$  is the predictive probability distribution and  $y'$  is the realizing observation (Gneiting and Raftery 2007). Under the class of kernel scores, which includes the Brier score and the continuous ranked probability score, an analogue of the Cover-Hart inequality applies, in that

$$\alpha \equiv \mathbb{E}_P S(P, Y') \leq \beta \equiv \mathbb{E}_P S(\delta_Y, Y') \leq 2\alpha, \quad (2)$$

where, again,  $Y$  and  $Y'$  are independent with distribution  $P$ , and  $\delta_Y$  is the point or Dirac random probability measure in  $Y$ . The paper closes with a discussion in Section 4, where we relate to the empirical success of reasoning and learning by analogy in general, and of nearest neighbor techniques in particular.

## 2 Point predictions based on a single observation

We now discuss the generality of the Cover-Hart inequality. Toward this end, we let  $\mathcal{P}$  be the family of the Radon probability measures on a Hausdorff space  $(\Omega, \mathcal{B})$ , where  $\mathcal{B}$  is the Borel- $\sigma$ -algebra. We say that a function  $L : \Omega \times \Omega \rightarrow \mathbb{R}$  is measurable if it is measurable with respect to either argument when the other argument is fixed.

**Definition 2.1.** The *Cover-Hart class* consists of the measurable functions  $L : \Omega \times \Omega \rightarrow [0, \infty)$  which are such that  $L(y, y) = 0$  for all  $y \in \Omega$ , and

$$\alpha \equiv \inf_y \mathbb{E}_P L(y, Y') \leq \beta \equiv \mathbb{E}_P L(Y, Y') \leq 2\alpha \quad (3)$$

for all probability measures  $P \in \mathcal{P}$ , where  $Y$  and  $Y'$  are independent with distribution  $P$ .

Under a loss function in the Cover-Hart class, half the information in an infinite sample is contained in a sample of size one, in the sense that predicting a future observation from a single past observation incurs at most twice the Bayes risk.

**Theorem 2.2.** *The Cover-Hart class is a convex cone.*

*Proof.* Suppose that  $L_1$  and  $L_2$  belong to the Cover-Hart class and let  $c_1, c_2 \geq 0$ . Then the convex combination  $L = c_1 L_1 + c_2 L_2$  is measurable,  $L(y, y) = 0$  for all  $y \in \Omega$ , and

$$\begin{aligned} \mathbb{E}_P L(Y, Y') &= c_1 \mathbb{E}_P L_1(Y, Y') + c_2 \mathbb{E}_P L_2(Y, Y') \\ &\leq 2c_1 \inf_{y_1} \mathbb{E}_P L_1(y_1, Y') + 2c_2 \inf_{y_2} \mathbb{E}_P L_2(y_2, Y') \\ &\leq 2 \inf_y (c_1 \mathbb{E}_P L_1(y, Y') + c_2 \mathbb{E}_P L_2(y, Y')) \\ &\leq 2 \inf_y \mathbb{E}_P L(y, Y') \end{aligned}$$

for every  $P \in \mathcal{P}$ , whence  $L$  belongs to the Cover-Hart class.  $\square$

The following result is based on a slight extension of an argument of Cover (1977), who implicitly assumed the existence of a Bayes rule.

**Theorem 2.3.** *Any measurable metric belongs to the Cover-Hart class.*

*Proof.* If  $L$  is a measurable metric, then  $L$  is nonnegative with  $L(y, y) = 0$  and  $L(y, y'') \leq L(y, y') + L(y', y'')$  for all  $y, y', y'' \in \Omega$ . Given any  $P \in \mathcal{P}$  there exists a sequence  $(y_n)$  in  $\Omega$  such that

$$\alpha = \inf_y \mathbb{E}_P L(y, Y') = \lim_{n \rightarrow \infty} \mathbb{E}_P L(y_n, Y').$$

Thus,

$$\alpha \leq \beta = \mathbb{E}_P L(Y, Y') \leq \mathbb{E}_P (L(Y, y_n) + L(y_n, Y')) = 2 \mathbb{E}_P L(y_n, Y')$$

for all integers  $n = 1, 2, \dots$ . The Cover-Hart inequality (3) emerges in the limit as  $n \rightarrow \infty$ , as desired.  $\square$

A function  $L : \Omega \times \Omega \rightarrow [0, \infty)$  is said to be a negative definite kernel if it is symmetric in its arguments, with  $L(y, y) = 0$  for all  $y \in \Omega$ , and

$$\sum_{i=1}^n \sum_{j=1}^n a_i a_j L(y_i, y_j) \leq 0$$

for all finite systems of points  $y_1, \dots, y_n \in \Omega$  and coefficients  $a_1, \dots, a_n \in \mathbb{R}$  such that  $a_1 + \dots + a_n = 0$ . Negative definite kernels play major roles in harmonic analysis (Berg, Christensen and Ressel 1984) and in the theory of stochastic processes, where they arise as the structure functions or variograms of random functions with stationary increments (Gneiting, Sasvári and Schlather 2002). A wealth of examples of such functions can be found in the monograph by Berg, Christensen and Ressel (1984) and the references therein.

**Theorem 2.4.** *Any continuous negative definite kernel belongs to the Cover-Hart class.*

*Proof.* If  $\mathbb{E}_P L(y, Y') = \infty$  for all  $y$ , then clearly the Cover-Hart inequality (3) holds. Thus, we may assume that  $\alpha = \inf_y \mathbb{E}_P L(y, Y')$  is finite. By Theorem 2.1 in Berg, Christensen and Ressel (1984, p. 235),

$$\mathbb{E}_P L(Y, Y') + \mathbb{E}_Q L(Z, Z') \leq 2 \mathbb{E}_{Q,P} L(Z, Y'),$$

where  $P$  and  $Q$  are Radon measures,  $Y$  and  $Y'$  have distribution  $P$ ,  $Z$  and  $Z'$  have distribution  $Q$ , and  $Y, Y', Z, Z'$  are independent. When  $Q$  is the point measure in  $y \in \Omega$ , the above inequality implies that  $\beta = \mathbb{E}_P L(Y, Y') \leq 2 \mathbb{E}_P L(y, Y')$  for all  $y$ , whence the Cover-Hart inequality is satisfied.  $\square$

We now discuss a few special cases, which are summarized in Table 1. If  $\Omega$  is a discrete space, the misclassification loss,  $L(y, y') = \mathbb{1}(y \neq y')$  is a continuous negative definite kernel. Thus, Theorem 2.4 applies and reduces to a classical result. When  $\Omega$  is finite, the upper bound in the inequality can be strengthened (Cover and Hart 1967).

Table 1: Examples of negative definite kernels. Here,  $\mathbb{Z}$  denotes the integers,  $\mathbb{R}$  the real numbers, and  $\mathbb{S}^{d-1}$  the unit sphere in the Euclidean space  $\mathbb{R}^d$ , where  $d \geq 2$ . The symbol  $\mathbb{1}(\cdot)$  stands for an indicator function,  $\|\cdot\|_p$  for the  $\ell_p$ -norm or quasi-norm in  $\mathbb{R}^d$ , and gcd for the geodetic or great circle distance on  $\mathbb{S}^{d-1}$ .

Space	Kernel	Parameters
$\mathbb{Z}$	$L(y, y') = \mathbb{1}(y \neq y')$	
$\mathbb{R}$	$L(y, y') =  y - y' ^q$	$q \in (0, 2]$
$\mathbb{R}^2$	$L(y, y') = \ y - y'\ _p^q$	$p \in (2, \infty], q \in (0, 1]$
$\mathbb{R}^d$	$L(y, y') = \ y - y'\ _p^q$	$p \in (0, 2], q \in (0, p]$
$\mathbb{S}^{d-1}$	$L(y, y') = \text{gcd}(y, y')$	

If  $\Omega$  is the real line,  $\mathbb{R}$ , the squared error loss function,  $L(y, y') = (y - y')^2$  is a continuous negative definite kernel. For a far-reaching generalization, let  $\|\cdot\|_p$  denote the standard  $\ell_p$ -norm or quasi-norm in the Euclidean space  $\mathbb{R}^d$ . Schoenberg's theorem (Schoenberg 1938; Berg, Christensen and Ressel 1984, p. 74) and a strand of literature culminating in the work of Koldobskiĭ (1992) and Zastavnyi (1993) demonstrate that the kernel

$$L(y, y') = \|y - y'\|_p^q$$

is negative definite under the conditions stated in Table 1, but not otherwise. Theorem 2.4 applies and the respective loss function is a member of the Cover-Hart class. To give an explicit example, if  $m = 1$  and the probability measure  $P$  is Gaussian, then  $\alpha = 2^{q/2}\beta$ .

Negative definite kernels can readily be constructed from positive definite functions (Schoenberg 1938; Gneiting, Sasvári and Schlather 2002). In this light, graph kernels (Borgwardt et al. 2005; Vishwanathan et al. 2010) and related types of positive definite functions on discrete structured spaces, as reviewed by Hofmann, Schölkopf and Smola (2008), yield Cover-Hart loss functions that are relevant to the prediction of highly structured objects, such as strings, trees, graphs and patterns.

### 3 Probabilistic predictions based on a single observation

Thus far, we have studied single-valued point forecasts. In this section, we turn to probabilistic predictions, where the forecasts take the form of predictive probability distributions over future quantities and events (Gneiting 2008). Technically, we retain the above setting and let  $\mathcal{P}$  denote the class of the Radon probability measures on a Hausdorff space  $(\Omega, \mathcal{B})$ . Predictive performance is evaluated by means of a score,

$$S(Q, y'),$$

that quantifies the loss when the probabilistic forecast is the Radon probability measure  $Q \in \mathcal{P}$ , and the realizing observation is  $y' \in \Omega$ .

A scoring rule thus is a function  $S : \mathcal{P} \times \Omega \rightarrow \mathbb{R}$ . It is called proper if

$$\mathbb{E}_P S(P, Y') \leq \mathbb{E}_P S(Q, Y')$$

for all probability measures  $P, Q \in \mathcal{P}$ , where  $Y'$  has distribution  $P$  and the expectations are assumed to exist. In other words, proper scoring rules encourage careful and honest probabilistic predictions and prevent hedging.

By Theorem 4 of Gneiting and Raftery (2007), proper scoring rules can be constructed from negative definite kernels, as follows.<sup>1</sup> Let  $L$  be a nonnegative, continuous negative definite kernel. Then the scoring rule

$$S(P, y') = \mathbb{E}_P L(Y, y') - \frac{1}{2} \mathbb{E}_P L(Y, Y')$$

is proper relative to the class of the Radon probability measures  $P$  for which the expectation  $\mathbb{E}_P L(Y, Y')$  is finite, where  $Y$  and  $Y'$  are independent with distribution  $P$ . Scoring rules of this form are referred to as kernel scores, and several of the most popular and most frequently used examples belong to this class, including both the Brier score and the continuous ranked probability score.

Under a kernel score, a straightforward calculation leads to a natural analogue of the Cover-Hart inequality that applies to probabilistic predictions. Specifically, if we define  $\alpha \equiv \inf_Q \mathbb{E}_P S(Q, Y')$  and  $S$  is a kernel score, a straightforward calculation shows that

$$\alpha = \mathbb{E}_P S(P, Y') \leq \beta \equiv \mathbb{E}_P S(\delta_Y, Y') = 2\alpha, \quad (4)$$

where  $Y$  and  $Y'$  are independent with distribution  $P$ , and  $\delta_Y$  is the point or Dirac random probability measure in  $Y$ . Again, half the information in an infinite sample is contained in a sample of size one, in that probabilistically predicting a future observation from a single past observation incurs at most twice the Bayes risk.

## 4 Discussion

Despite being well known in pattern analysis and information theory (see, for example, Devroye, Györfi and Lugosi 1996), the ground breaking work of Cover and Hart (1967) and Cover (1968) has hardly received any attention in the statistical literature.

In this paper, we have demonstrated that the Cover-Hart inequality (1) applies whenever the loss function is a measurable metric, or a continuous negative definite kernel. Many but not all metrics are negative definite (Meckes 2011), and so the two families may have distinct members. An interesting open question is whether or not the Cover-Hart class

---

<sup>1</sup>As pointed out to the author by Jochen Fiedler, Theorem 4 of Gneiting and Raftery (2007) ought to be formulated relative to the class of the Radon probability measures on  $\Omega$ , as opposed to the larger class of the Borel probability measures. While we are unaware of a counterexample for Borel measures, the result of Berg, Christensen and Ressel (1984) used in the proof of the theorem applies to Radon measures only.

equals the convex cone that is generated by these two families. In particular, I do not know whether or not the Cover-Hart class contains any asymmetric loss functions.

Typically, predictions are conditional on an information set, leading to natural ramifications of single nearest neighbor methods, such as nonparametric regression (Stone 1977) and kernel estimators of conditional predictive distributions (Hyndman, Bashtannyk and Grunwald 1996; Hall, Wolff and Yao 1999). While we have suppressed the dependence on the information set in our work, the Cover-Hart inequality remains valid in this setting, by conditioning on and integrating over the information set.

In this light, if empirically observed mean score differentials exceed 100%, this may suggest that forecasters have distinct information sets. A simulation example is reported on in Tables 4 and 6 of Gneiting (2011), where the differences in the predictive scores between Mr. Bayes and his competitors, whose predictions are based on thoroughly distinct information sets, are striking.

From an applied perspective, the Cover-Hart inequality (1) for point forecasts, and its analogue (2) for probabilistic forecasts, allow for interesting interpretations. Given that under many of the most prevalent loss functions used in practice, Alice, despite having an infinite sample at her disposal, can at most halve Bob’s risk, who has access to a sample of size one only, it is not surprising that empirically observed differentials in the predictive performance of competing forecasters tend to be small. For example, this was observed in the Netflix contest, where predictive performance was measured in terms of the (root mean) squared error (Feuerverger, He and Khatri 2012). Taking a much broader perspective, the Cover-Hart inequality may contribute to our understanding of the empirical success not only of nearest neighbor techniques and their ramifications, but reasoning and learning by analogy in general (Gentner and Holyoak 1997).

## Acknowledgements

The author is grateful to the Alfried Krupp von und zu Behlen Foundation for support, and thanks Jochen Fiedler, Marc Genton, Christoph Schnörr and Jon Wellner for discussions and references.

## References

- Berg, C., Christensen, J. P. R., and Ressel, P. (1984). *Harmonic Analysis on Semigroups*. New York: Springer.
- Borgwardt, K. M., Ong, C. S., Schönauer, S., Vishwanathan, S. V. N., Smola, A. J. and Kriegel, H.-P. (2005). Protein function prediction via graph kernels. *Bioinformatics* 21, i47–i56.
- Cover, T. M. (1968). Estimation by the nearest neighbor rule. *IEEE Transactions on Information Theory*, 14, 50–55.

- Cover, T. M. (1977). Comment. *Annals of Statistics*, 5, 627–628.
- Cover, T. M. and Hart, P. E. (1967). Nearest neighbor pattern classification. *IEEE Transactions on Information Theory*, 13, 21–27.
- Devroye, L., Györfi, L., and Lugosi, G. (1996). A Probabilistic Theory of Pattern Recognition. Springer, New York.
- Feuerverger, A., He, Y. and Khatri, S. (2012). Statistical significance of the Netflix challenge. *Statistical Science*, in press.
- Gentner, D. and Holyoak, K. J. (1997). Reasoning and learning by analogy: Introduction. *American Psychologist*, 52, 32–34.
- Gneiting, T. (2008). Editorial: Probabilistic forecasting. *Journal of the Royal Statistical Society Series A: Statistics in Society*, 171, 319–321.
- Gneiting, T. (2011). Making and evaluating point forecasts. *Journal of the American Statistical Association*, 106, 746–762.
- Gneiting, T. and Raftery, A. E. (2007). Strictly proper scoring rules, prediction, and estimation. *Journal of the American Statistical Association*, 102, 359–378.
- Gneiting, T., Sasvári, Z. and Schlather, M. (2001). Analogies and correspondences between variograms and covariance functions. *Advances in Applied Probability*, 33, 617–630.
- Hall, P., Wolff, R. C. L. and Yao, Q. (1999). Methods for estimating a conditional distribution function. *Journal of the American Statistical Association*, 94, 154–163.
- Hofmann, T., Schölkopf, B. and Smola, A. (2008). Kernel methods in machine learning. *Annals of Statistics*, 36, 1171–1220.
- Hyndman, R. J., Bashtannyk, D. M. and Grunwald, G. K. (1996). Estimating and visualizing conditional densities. *Journal of Computational and Graphical Statistics*, 5, 315–336.
- Koldobskii, A. L. (1992). Schoenberg’s problem on positive definite functions. *St. Petersburg Mathematical Journal*, 3, 563–570.
- Meckes, M. W. (2011). Positive definite metric spaces. Preprint, [arxiv:1012.5863v3.pdf](#).
- Schoenberg, I. J. (1938). Metric spaces and positive definite functions. *Transactions of the American Mathematical Society*, 44, 522–536.
- Stone, C. J. (1977). Consistent nonparametric regression (with discussion). *Annals of Statistics*, 5, 595–645.
- Vishwanathan, S. V. N., Schraudolph, N., Kondor, R., and Borgwardt, K. M. (2010). Graph kernels. *Journal of Machine Learning Research* 11, 1201–1242.
- Zastavnyi, V. P. (1993). Positive definite functions depending on the norm. *Russian Journal of Mathematical Physics*, 1, 511–522.